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Electron transport models with diffusion gradient and electric field, using the maximum anisotropic approximation

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Abstract. The maximum anisotropic approximation for electron transport in an electric field is extended to the position-dependent case. Models utilising non-polar optical, acoustic and piezoelectric phonon scattering processes are constructed. The piezoelectric model produces the Euler–Darboux equation in the position-dependent case and an analytical solution is given. Analytical solutions are given for each of the models in the position-independent case.

1. Introduction

In electron transport problems in the presence of a DC electrical field it is often a good approximation to assume a nearly isotropic form for the probability distribution function in momentum or energy space—the so-called nearly isotropic approximation (NIA) [1, 2, 4]. In this approximation the field modifies the distribution function by a small anisotropic term, dependent on the energy and the angle, θ, between the electron momentum vector and the field direction. Essentially, this approximation is the first-order truncation of the Legendre expansion of the distribution function in the angle θ, about the direction of the field. If f_0, f_1 denote the isotropic and anisotropic terms in the NIA, then substitution in the Boltzmann transport equation (BTE) and use of the orthogonality properties of the Legendre polynomials results in two coupled equations for f_0, f_1. Under certain further assumptions f_1 may be eliminated and a single equation obtained for f_0. Much use of this method has been made in electron transport problems, particularly in the position-independent case, in which the equation for f_0 reduces to a second-order ordinary differential equation in energy space which is often analytically solvable [2]. In the case of inhomogeneous transport, in which the distribution function is spatially dependent and spatial boundary conditions are relevant, the equation for f_0 becomes a second-order partial differential equation in position and energy variables, which again may be analytically solvable in certain circumstances [1, 3–5].

Many years ago Baraff [6] drew attention to an approximation at the other extreme to the NIA, i.e. when the distribution function is 'maximally' distorted due to a preponderance of electrons streaming in the field direction. Baraff argued for an alternative form of truncation of the Legendre expansion in this case which he called the 'maximum anisotropy approximation' (MAA). This method essentially uses an extremely spiked dependence of the distribution function on the angular variable θ to truncate the Legendre expansion in a maximally anisotropic manner. Applied to second order in the Legendre series it again produces coupled equations for f_0 and f_1, as in the NIA case, but of course the equations are different. Baraff applied the method to spatially-independent electron transport in a strong uniform electric field in a semiconductor with optical-phonon-dominated scattering in the presence of ionisation processes. The resulting equations for his equivalents to f_0 and f_1 are a coupled system of first-order ordinary differential equations in energy space, which he solved by a number of approximations. The method has also been applied to other problems [7].

In this paper we generalise Baraff's treatment to the spatially-dependent case, analogous to that considered for the NIA in [1, 3, 4, 5], and also extend the analysis to other scattering processes. The coupled equations for f_0 and f_1 are now complicated first-order partial differential equations in position and energy space. We examine to what extent the elimination of f_1 from these equations produces a tractable equation for f_0 for different scattering processes. Without some approximations in the physical parameters such as scattering rates and electric field, the resulting equations are not tractable analytically for the cases of non-polar optical scattering or acoustic phonon scattering. However, the piezoelectric case is interesting in that it may be reduced to the well known Euler–Darboux equation [8, 9], for which a formal analytic solution may be obtained. However, applying the physical boundary conditions may present some difficulties. We also look at the position-independent models (analogous to that of Baraff) for acoustic and piezoelectric scattering and find that solutions are possible in both cases, just as in the ionisation-free optical phonon case. It is not our intention to examine the detailed physical properties of these models and solutions here, but only to study the extent to which tractable analytical solutions exist for the various models.

2. The maximum anisotropy approximation

For the case of one-dimensional electron transport in a semiconductor, in the presence of a uniform field, ζ, and a concentration gradient along the x axis, the probability distribution function $f(x, k, t)$ in the quasiclassical effective mass approximation, in the steady state, satisfies the BTE

$$(\hbar k_x/M_e^*)(\partial f/\partial x) + (e\zeta/\hbar)(\partial f/\partial k_x) = C(f). \tag{2.1}$$

Here $k_x = k\cos\theta$ is the component of the wave vector in the field direction, with which it makes an angle θ. M_e^* is the effective mass and $C(f)$ is the collision integral.

Assuming, as is usual, that the only form of anisotropy in wave vector space is the electric field, we can expand f in a Legendre series about the direction of the field, i.e. in the form

$$f(x, k_x) \equiv f(x, k, \theta) = \sum_{l=0}^{\infty} f_l(x, E)P_l(\cos\theta) \tag{2.2}$$

where E, the energy, is given by $E = \hbar^2k^2/2M_e^*$. Note that the expansion coefficients in (2.2), $f_l(x, E)$, are not quite the same as those used in the usual approach to the NIA [2, 4]. There it is usual to assume an expansion of the form

$$f(x, k_x) = f_0(x, E) + kf_1(x, E)P_1(\cos\theta)$$

with the factor k in the second term. This form of the expansion, while convenient for the NIA does not generalise easily to the full Legendre approximation, so here we retain the form (2.2). Thus, the f_1 we use in this paper is k times the f_1 used in [4]. The f_1 used in [1] on the other hand is equivalent to that used here.

Analogous to (2.2), the collision integral in (2.1) may be expanded in the form

$$C(f) = \sum_{l=0}^{\infty} C_l(x, E)P_l(\theta) \tag{2.3}$$

where the $C_l(x, E)$ are in fact functions of the f_l and, on the usual assumption of virtually isotropic scattering, only C_0 and C_1 make significant contributions.

To facilitate the substitution of the Legendre expansion in (2.1) we introduce k-space polar coordinates in the form

$$\begin{aligned} E &= \hbar^2k^2/2M_e^* = (\hbar^2/2M_e^*)(k_x^2 + k_y^2 + k_z^2) \\ \mu &= \cos\theta = k_x/k. \end{aligned} \tag{2.4}$$

In terms of these coordinates, the BTE operator in (2.1) becomes

$$\frac{\hbar k_x}{M_e^*}\frac{\partial f}{\partial x} + \frac{e\zeta}{\hbar}\frac{\partial f}{\partial k_x} \equiv \frac{\hbar k\mu}{M_e^*}\frac{\partial f}{\partial x} + \frac{e\zeta}{\hbar}\left(\frac{\hbar\sqrt{2}}{M_e^{*1/2}}E^{1/2}\mu\frac{\partial f}{\partial E} + \frac{\hbar}{\sqrt{2}M_e^{*1/2}}E^{-1/2}(1-\mu^2)\frac{\partial F}{\partial\mu}\right). \tag{2.5}$$

Substituting (2.2) into the right hand side of (2.5) and using the well known identities:

$$\mu P_n(\mu) = [1/(2n+1)][(n+1)P_{n+1}(\mu) + nP_{n-1}(\mu)] \tag{2.6}$$

$$(1-\mu^2)\,\partial P_n(\mu)/\partial\mu = [n(n+1)/(2n+1)](P_{n-1} - P_{n+1}) \tag{2.7}$$

we obtain, on equating the coefficients of $P_n(\mu)$ with (2.3):

$$\begin{aligned} &(1/\sqrt{2M_e^*E})[\tfrac{2}{3}EDf_1 + (2e\zeta/3)f_1] = C_0 \\ &(1/\sqrt{2M_e^*E})(2EDf_0 + \tfrac{4}{5}EDf_2 + \tfrac{6}{5}e\zeta f_2) = C_1 \\ &(1/\sqrt{2M_e^*E})[\tfrac{4}{3}EDf_1 + \tfrac{6}{7}EDf_3 + \tfrac{12}{7}e\zeta f_3 - (2e\zeta/3)f_1] = C_2 \end{aligned} \tag{2.8}$$

etc, where $D \equiv \partial/\partial x + e\zeta(\partial/\partial E)$.

Apart from notation, these are the equations given by Baraff. Formally, one obtains the collision integral coefficients, C_n, by substituting the Legendre expansion in the collision integral

$$C(f) = \frac{-V}{(2\pi)^3}\int [P(\boldsymbol{k}, \boldsymbol{k}')f(\boldsymbol{k}) - P(\boldsymbol{k}', \boldsymbol{k})f(\boldsymbol{k}')]\,\mathrm{d}\boldsymbol{k}' \tag{2.9}$$

choosing the transition probability $P(\boldsymbol{kk}')$ appropriate to the collision process of interest and making various plausible approximations to produce expressions for the C_n of practical use. In his original paper Baraff adopted a direct physical approach to deduce the C_0 and C_1 which he required (S_0 and S_1 in his notation) for his model of optical phonon scattering and ionisation processes. Here we will be more general and consider forms of C_0 and C_1 appropriate to some standard models for a wide range of scattering processes [2]—essentially those for which a relaxation approximation for f_1 is permissible. First, however, we describe Baraff's truncation procedure for the maximum anisotropic approximation.

In the NIA, the set of equations (2.8) is truncated after the first two terms in the Legendre series, leaving only the first two coupled equations for f_0 and f_1 with f_2 put equal to zero. Then if one can assume a relaxation form for C_1, taking it proportional (up to an energy dependent factor) to f_1, we can eliminate f_1 from the first two equations and obtain a single equation for f_0. In the MAA, on the other hand, it is assumed that f is so highly peaked in the field direction that it may be assumed to have the form

$$f(x, E, \mu) = g(x, E)\delta(1 - \mu). \tag{2.10}$$

Then the Legendre coefficients will have the form

$$f_n(x, E) = \frac{2n+1}{2}\int_{-1}^{1} g(x, E)\delta(1-\mu)P_n(\mu)\,\mathrm{d}\mu = [(2n+1)/2]g(x, E)P_n(1)$$

from which we deduce, since $P_n(1) = 1$ for all n, that

$$f_n(x, E) = [(2n+1)/(2n-1)]f_{n-1}(x, E). \tag{2.11}$$

We now truncate the Legendre series at the Nth term by assuming

$$f_N(x, E) = [(2N+1)/(2N-1)]f_{N-1}(x, E) \tag{2.12}$$

which is the basis for the MAA. For its general region of validity and its consistency with the NIA in the isotropic limit see [6].

Applying the above truncation to the case of $N = 2$, i.e. dropping the equations for f_2 and higher-order coefficients, the scheme (2.8) produces the following equations for f_0 and f_1 on use of (2.12)

$$(1/\sqrt{2M_e^* E})(\tfrac{4}{3}EDf_1 + (2e\zeta/3)f_1) = C_0 \tag{2.13}$$

$$(2/\sqrt{M_e^* E})(EDf_0 + \tfrac{2}{3}EDf_1 + e\zeta f_1) = C_1 \tag{2.14}$$

and it is these coupled equations which we have to solve for f_0 and f_1. For this we need the expressions for the collision integral coefficients C_0 and C_1, which of course depend on the scattering process under consideration. The forms used for C_0 and C_1 are notoriously subject to approximations and assumptions which may be questionable in such extreme cases as high field and the presence of carrier streaming. However, we will here adopt the pragmatic attitude that the field does not affect the scattering mechanism and will consider the standard models of randomising or elastic collisions for first order in phonon energy. The standard reference here is [2], to which the reader is referred for further discussion of the results which we use. As noted earlier, the f_1 used in the present paper is k times that used in [2] and bearing this in mind we quote the results for C_0 and C_1 for the different scattering mechanisms, to first order in phonon energy.

In each of the cases of non-polar optical, acoustic and piezoelectric phonon scattering we can obtain a relaxation approximation for C_1

$$C_1 = -f_1(E)/\tau_1(E)$$

where the relaxation time τ_1 takes the form given below, along with the C_0 expressions to first order in phonon energy.

Non-polar optical phonon scattering:

$$\tau_1 = \tau_{\mathrm{opm}}E^{-1/2} \tag{2.15}$$

$$C_0 = (1/\tau_{\mathrm{ope}})E^{-1/2}(f_0 + E(\partial f_0/\partial E)) = (1/\tau_{\mathrm{ope}})E^{-1/2}(\partial(Ef_0)/\partial E). \tag{2.16}$$

Acoustic phonon scattering:

$$\tau_1 = \tau_{\mathrm{acm}}E^{-1/2} \tag{2.17}$$

$$C_0 = (1/\tau_{\mathrm{ace}})E^{1/2}(2f_0 + E(\partial f_0/\partial E)) = (1/\tau_{\mathrm{ace}})E^{-1/2}\,\partial(E^2 f_0)/\partial E. \tag{2.18}$$

Piezoelectric phonon scattering:

$$\tau_1 = \tau_{\rm pzm} E^{1/2} \tag{2.19}$$

$$C_0 = (1/\tau_{\rm pze})E^{-1/2}(f_0 + E\,\partial f_0/\partial E) = (1/\tau_{\rm pze})E^{-1/2}\,\partial(Ef_0)/\partial E. \tag{2.20}$$

The $\tau_{\rm opm}$, $\tau_{\rm acm}$, and $\tau_{\rm pzm}$ are the respective C_1 relaxation time constants for non-polar optical phonon, acoustic and piezoelectric phonon scattering. $\tau_{\rm ope}$, $\tau_{\rm ace}$, and $\tau_{\rm pze}$ denote the corresponding time constants associated with the symmetric collision term C_0 [2].

We now examine the maximum anisotropy approximation equations for each of these scattering processes.

3. Non-polar optical phonon scattering

In the x-independent case this is essentially the situation modelled by Baraff [6]. Using (2.15) and (2.16) in (2.13) and (2.14) gives

$$\tfrac{2}{3}EDf_1 + (2\lambda/3)f_1 = \alpha_0\,\partial(Ef_0)/\partial E \tag{3.1}$$

$$EDf_0 + \tfrac{2}{3}EDf_1 + \lambda f_1 = -(E/\beta_0)f_1 \tag{3.2}$$

where

$$\alpha_0 = \sqrt{2M_e^*}/\tau_{\rm ope} \qquad \beta_0 = \tau_{\rm opm}/\sqrt{2M_e^*} \qquad \lambda = e\zeta. \tag{3.3}$$

We may now eliminte f_1 from these two equations by subtracting (3.1) from (3.2) to eliminate Df_1, leaving an expression for f_1, in terms of f_0 and Df_0, which may then be substituted back into (3.1). The rather involved calculations result in the following equation

$$\begin{aligned}E(E/\beta_0 + \lambda/3)[\partial^2\psi/\partial x^2 + (2\lambda + \alpha_0)\,\partial^2\psi/\partial x\,\partial E + (\lambda^2 + \lambda\alpha_0)\,\partial^2\psi/\partial E^2] \\ + [(3\alpha_0/2\beta_0^2)E^2 - (\lambda/\beta_0)(\lambda - \alpha_0)E + \alpha_0\lambda^2/2]\partial\psi/\partial E \\ - (\lambda/\beta_0)E(\partial\psi/\partial x) + (\lambda^2/\beta_0)\psi = 0\end{aligned} \tag{3.4}$$

where $\psi = Ef_0$. This is a hyperbolic second-order partial differential equation for ψ. The roots of its auxiliary equation are $-\lambda$ and $-(\lambda + \alpha_0)$, so it may be converted to canonical form by choosing coordinates

$$\xi = E - \lambda x \qquad \eta = E - (\lambda + \alpha_0)x.$$

However, neither (3.4) or its canonical form lend themselves to a practicable analytical approach, so we will move on to the case of position-independent transport. In this case, dropping the spatial derivatives in (3.1) and (3.2) gives

$$(2\lambda/3)E\,\mathrm{d}f_1/\mathrm{d}E + (2\lambda/3)f_1 = \alpha_0\,\mathrm{d}(Ef_0)/\mathrm{d}E \tag{3.5}$$

$$\lambda E\,\mathrm{d}f_0/\mathrm{d}E + \tfrac{2}{3}EDf_1 + \lambda f_1 = -(E/\beta_0)f_1. \tag{3.6}$$

On eliminating f_1 from (3.5) and (3.6) we obtain, of course, the position-independent form of (3.4):

$$\begin{aligned}\lambda(\lambda + \alpha_0)E(E/\beta_0 + \lambda/3)\,\mathrm{d}^2\psi/\mathrm{d}E^2 + \{(2\alpha_0/2\beta_0^2)E^2 \\ - [\lambda(\lambda - \alpha_0)/\beta_0]E + \alpha_0\lambda^2/2\}\,\mathrm{d}\psi/\mathrm{d}E + (\lambda^2/\beta_0)\psi = 0.\end{aligned} \tag{3.7}$$

It is interesting to compare (3.5) and (3.6) with the corresponding equations of Baraff [6]. Putting $m_1 = Ef_1$ and $m_0 = Ef_0 = \psi$, (3.5) and (3.6) become

$$\mathrm{d}m_1/\mathrm{d}E = (3\alpha_0/2\lambda)\,\mathrm{d}m_0/\mathrm{d}E \tag{3.8}$$

$$(2\lambda\beta_0/3)\,\mathrm{d}m_1/\mathrm{d}E + (1+\lambda\beta_0/3E)m_1 + \beta_0\lambda\,\mathrm{d}m_0/\mathrm{d}E - (\lambda\beta_0/E)m_0 = 0. \tag{3.9}$$

These are to be compared with Baraff's equations

$$(Q/3)\,\mathrm{d}m_1/\mathrm{d}E + rm_0 - (1-r)E_R\,\mathrm{d}m_0/\mathrm{d}E = 0 \tag{3.10}$$

$$(2Q/3)\,\mathrm{d}m_1/\mathrm{d}E + (1+Q/3E)m_1 + Q\,\mathrm{d}m_0/\mathrm{d}E - (Q/E)m_0 = 0 \tag{3.11}$$

which include an ionisation process ($r \neq 0$) in addition to an optical phonon scattering model. Removing the ionisation by setting $r = 0$ allows direct comparison with (3.8) and (3.9) from which we obtain the connection between Baraff's parameters and our own:

$$Q = \lambda\beta_0 \qquad E_R = \alpha_0\beta_0/2 \tag{3.12}$$

(Baraff has fewer parameters because he assumed the same relaxation times for both C_0 and C_1, whereas we have not assumed $\tau_{\mathrm{opm}} = \tau_{\mathrm{ope}}$).

As did Baraff in the $r = 0$ case, we can solve (3.8) and (3.9) analytically. Thus (3.8) integrates directly to

$$m_0 = (2\lambda/3\alpha_0)m_1 + A \tag{3.13}$$

with A an arbitrary constant. Then substitution into (3.9) gives

$$\mathrm{d}m_1/\mathrm{d}E + (b + a/E)m_1 = c/E \tag{3.14}$$

where

$$a = (\alpha_0 - 2\lambda)/2(\alpha_0+\lambda) \qquad b = 3\alpha_0/2\lambda\beta_0(\alpha_0+\lambda) \qquad c = 3\alpha_0 A/2(\alpha_0+\lambda). \tag{3.15}$$

(3.14) has the general solution

$$m_1(E) = \frac{c}{a}\left(1 + bE^{-a}\,\mathrm{e}^{-bE}\int_E^{E_i} t^a\,\mathrm{e}^{bt}\,\mathrm{d}t - (E/E_i)^{-a}\,\mathrm{e}^{b(E_i-E)}\right) + m_1(E_i)(E/E_i)^{-a}\,\mathrm{e}^{b(E_i-E)} \tag{3.16}$$

which is Baraff's equation, where E_i is some threshold energy below which no ionisation occurs. m_0 then follows from (3.13). We will not pursue further the physical implications of (3.16), which have been thoroughly discussed by Baraff.

4. Acoustic phonon scattering

Substituting (2.17) and (2.18) into (2.13) and (2.14) gives

$$\tfrac{2}{3}EDf_1 + (2\lambda/3)f_1 = \alpha_{\mathrm{a}}\,\partial(E^2 f_0)/\partial E \tag{4.1}$$

$$EDf_0 + \tfrac{2}{3}EDf_1 + \lambda f_1 = -Ef_1/\beta_{\mathrm{a}} \tag{4.2}$$

where

$$\alpha_{\mathrm{a}} = \sqrt{2M_{\mathrm{e}}^*}/\tau_{\mathrm{ace}} \qquad \beta_{\mathrm{a}} = 2\tau_{\mathrm{acm}}/\sqrt{2M_{\mathrm{e}}^*}.$$

Apart from the different form for C_0, these are identical to the polar optical case and similar, even more involved, calculations eliminating f_1 result in the following equation for $\psi = Ef_0$

$$E(E/\beta_a + \lambda/3)[\partial^2\psi/\partial x^2 + (2\lambda + \alpha_a E)\,\partial^2\psi/\partial x\,\partial E + \lambda(\lambda + \alpha_a E)\,\partial^2\psi/\partial E^2]$$
$$+ [(3\alpha_a/2\beta_a^2)E^3 + (3\lambda\alpha_a/\beta_a)E^2 + \lambda^2(7\alpha_a/6 - 1/\beta_a)E]\,\partial\psi/\partial x$$
$$+ (\alpha_a/\beta_a)E^2 + \lambda(\alpha_a/3 - 1/\beta_a)E]\,\partial\psi/\partial x$$
$$+ [(3\alpha_a/2\beta_a^2)E^2 + (\lambda\alpha_a/\beta_a)E + \lambda^2(1/\beta_a + \alpha_a/2)]\psi = 0. \tag{4.3}$$

Again, this is a hyperbolic partial differential equation. The roots of its auxiliary equation are $-\lambda$ and $-(\lambda + \alpha_a E)$, so it may be converted to canonical form by the choice of coordinates

$$\xi = E - \lambda x \qquad \eta = \ln(\lambda + \alpha_a E) - \alpha_a x.$$

While the general form (4.3) offers no simple solution, the position-independent case is more tractable. The equations (4.1) and (4.2) may be rewritten as

$$\tfrac{4}{3}\lambda\,\mathrm{d}m_1/\mathrm{d}E = \alpha_a\,\mathrm{d}(Em_0)/\mathrm{d}E \tag{4.4}$$

$$\lambda\,\mathrm{d}m_0/\mathrm{d}E + (2\lambda/3)\,\mathrm{d}m_1/\mathrm{d}E - (\lambda/E)m_0 + (\lambda/3E + 1/\beta_a)m_1 = 0 \tag{4.5}$$

with $m_0 = \psi = Ef_0$ and $m_1 = Ef_1$. From (4.4) we have

$$m_1 = (3\alpha_a/2\lambda)Em_0 + A \qquad (A = \text{constant})$$

which, substituting into (4.5) gives

$$\mathrm{d}m_0/\mathrm{d}E + [(3\alpha_a E^2 + 3\lambda\alpha_a\beta_a E - 2\lambda^2\beta_a)/2\lambda\beta_a E(\lambda + \alpha_a E)]m_0$$
$$= -A(\lambda\beta_a + 3E)/3\beta_a E(\lambda + \alpha_a E).$$

This linear first-order equation may be solved to give

$$m_0(E) = \frac{E\exp(-3E/2\lambda\beta_a)}{(\alpha_a E + \lambda)^{(5\alpha_a\beta_a - 3)/2\alpha_a\beta_a}}\left(\frac{m_0(0)}{\lambda^{(3-5\alpha_a\beta_a)/2\alpha_a\beta_a}}\right.$$
$$\left. - \frac{A}{3\beta_a}\int_0^E \frac{(\lambda\beta_a + 3x)\exp[3x/2\lambda\beta_a]\,\mathrm{d}x}{x^2(\alpha_a x + \lambda)^{3(1-\alpha_a\beta_a)/2\alpha^a\beta^a}}\right) \tag{4.6}$$

where $m_0(0)$ and A may be determined from the physical boundary conditions. (4.6) provides us with a model for homogeneous acoustic phonon scattering analogous to that provided by Baraff for optical phonon scattering.

5. Piezoelectric scattering

In this case the coupled equations for f_0 and f_1 become

$$\tfrac{4}{3}EDf_1 + (2\lambda/3)f_1 = \alpha_p\,\partial(Ef_0)/\partial E \tag{5.1}$$

$$EDf_0 + \tfrac{2}{3}EDf_1 + \lambda f_1 = -f_1/\beta_p \tag{5.2}$$

where

$$\alpha_p = \sqrt{2M_e^*}/\tau_{\mathrm{pze}} \qquad \beta_p = 2\tau_{\mathrm{pzm}}/\sqrt{2M_e^*}. \tag{5.3}$$

The calculations to eliminate f_1 are less strenous in this case and yield the equation

$$E[\partial^2\psi/\partial x^2 + (2\lambda + \alpha_p)\,\partial^2\psi/\partial x\,\partial E + \lambda(\lambda + \alpha_p)\,\partial^2\psi/\partial E^2]$$
$$+ (3\alpha_p/2)(\lambda + 1/\beta_p)\,\partial\psi/\partial E = 0 \tag{5.4}$$

with

$$\psi = EF_0. \tag{5.5}$$

Again, this is a hyperbolic equation which can be converted to canonical form by choosing coordinates

$$\xi = E - \lambda x \qquad \eta = E - (\lambda + \alpha_p)x. \tag{5.6}$$

The result is

$$\partial^2\psi/\partial\xi\,\partial\eta + \{A/[\lambda\eta - (\alpha_p + \lambda)\xi]\}(\partial\psi/\partial\xi + \partial\psi/\partial\eta) = 0 \tag{5.7}$$

where $A = (3\alpha_p^2/2\beta_p)(\beta_p\lambda + 1)/(2\lambda + \alpha_p)^2$. Finally, substitute

$$u = (\alpha_p + \lambda)\xi \qquad v = \lambda\eta \tag{5.8}$$

into (5.7) to obtain the equation

$$\partial^2\psi/\partial u\,\partial v - [\alpha/(u - v)]\,\partial\psi/\partial u - [\beta/(u - v)]\,\partial\psi/\partial v = 0 \tag{5.9}$$

where $\alpha = a/\lambda \geqslant 0$ and $\beta = a/(\lambda + \alpha_p) \geqslant 0$. (5.9) is a form of the Euler–Darboux equation [8, 9]. It may be shown that (5.9) has a general solution of the form

$$\psi = \int_v^u \varphi(\xi)(u - \xi)^\beta(\xi - v)^{-\alpha}\,d\xi + (u - v)^{1+\beta-\alpha}\int_v^u \theta(\xi)(u - \xi)^{\alpha-1}(\xi - v)^{-(1+\beta)}\,d\xi \tag{5.10}$$

where $\varphi(\xi)$ and $\theta(\xi)$ are arbitrary functions.

Substituting

$$\xi = tu + (1 - t)v = (u - v)t + v \qquad 0 \leqslant t \leqslant 1$$

the general solution (5.10) takes the form

$$\psi = (u - v)^{1+\beta-\alpha}\int_0^1 \varphi((u - v)t + v)t^\beta(1 - t)^{-\alpha}\,dt + \int_0^1 \theta((u - v)t + v)t^{\alpha-1}(1 - t)^{-(1+\beta)}\,dt$$

whence substituting back from (5.6) finally yields the solution of (5.4) in the form

$$\psi(x, E) = (\alpha_p E)^{1+\beta-\alpha}\int_0^1 \varphi(\alpha_p Et + \lambda E - \lambda(\lambda + \alpha_p)x)t^\beta(1 - t)^{-\alpha}\,dt + \int_0^1 \theta(\alpha_p Et + \lambda E - \lambda(\lambda + \alpha_p)x)t^{\alpha+1}(1 - t)^{-(1+\beta)}\,dt. \tag{5.11}$$

While such general analytical solutions are a luxury in models of this sort, enthusiasm for it is tempered by the obvious difficulty of determining the functions φ and θ appropriate to realistic boundary conditions. This may involve us in the solution of complicated integral equations, but at least the results will allow a closed form analytic solution.

The position-independent case for piezoelectric scattering is much simpler. (5.4) reduces to

$$\lambda(\lambda + \alpha_p)\, d^2\psi/dE^2 + (\alpha_p a/E)\, d\psi/dE = 0 \tag{5.12}$$

which is easily integrated to

$$\psi = \begin{cases} AE^{1-b}/(1-b) + B & \text{if } b \neq 1 \\ A \ln E + B & \text{if } b = 1 \end{cases} \tag{5.13}$$

where A and B are constants and $b = \alpha_p a/\lambda(\lambda + \alpha_p)$.

(5.13) gives a simple model for piezoelectric scattering in the position-independent case under conditions of streaming under a strong electric field. It generalises Baraff's analysis to piezoelectric scattering.

6. Conclusions

We have investigated the types of mathematical models of transport in an electric field with various types of scattering, using the maximum anisotropic approximation of Baraff. In the position-dependent case the non-polar optical and acoustic cases do not appear to be amenable to practicable analytical treatments, but the piezoelectric case reduces to the well known Euler–Darboux equation, of which it is an interesting example. An analytic solution for the piezoelectric case is given, although the application to real physical situations may be restricted by the difficulties of fitting the boundary conditions.

In the spatially-independent case, analytical solutions for all models are given and the application to physical situations should follow essentially the treatment of the non-polar optical case given by Baraff [6].

Such analytical models considered here are useful as first approximations in the modelling of specific devices or processes and also as test cases for the development of computer codes.

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